CS522 - Black-Scholes Formulas

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0.1 The Valuation of European Calls

Let us recall the distribution we obtained in the limit for a binomial model under the equivalent probabilities q:[1]

$$\begin{aligned} r_T &= \left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}N(0,1) \\ S(T) &= S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}N(0,1)\right]. \end{aligned}$$

where $N(0,1)$ denotes a standard normal random variable. From now on, we will use the notation z to refer to such a variable.

Based on the discrete examples that we have examined, we have concluded that for a given European payoff X, its time-0 value $V(0)$ is given by

$$V(0) = e^{-rT}\mathbb{E}_q[X(T)].$$

Without further proof, we will accept that the previous relation also holds in the limit, when the number of intervals into which with divide interval $[0, T]$ tends to infinity (or, alternatively, Δ tends to 0).

In the following, we will concentrate on the valuation of a European call with the payoff $X(T) = \max(S(T) - K, 0)$, where K is the strike price of the call.

We introduce the following notations:

$$\begin{aligned} \Phi(z) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}u^2}du \\ \varphi(z) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}. \end{aligned}$$

It is clear that Φ and φ are the c.d.f. and p.d.f., respectively, of a standard normal random variable.[2] Further, we have that $\Phi'(z) = \varphi(z)$.

We now directly compute the discounted expected value of the payoff of a European call:

$$\begin{aligned} V(0) &= e^{-rT}\int_{-\infty}^{+\infty} \max(S(T) - K, 0)\varphi(z)dz \\ &= e^{-rT}\int_{-\infty}^{+\infty} \max(S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}z\right] - K, 0)\varphi(z)dz. \end{aligned}$$

[1] See the lectures notes titled "Option Pricing: Building the Lattice (2)" for details.

[2] You can use Matlab functions *normcdf* and *normpdf*, respectively, to compute the values of these functions.

Let z_* be the value that satisfies the equation

$$S(T) = S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}z_*\right] = K.$$

We immediately get that

$$z_* = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{K}{S(0)} - \left(r - \frac{1}{2}\sigma^2\right)T\right].$$

We can now eliminate the max function (why?):

$$\begin{aligned}
V(0) &= e^{-rT}\int_{z_*}^{+\infty} \max(S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}z\right] - K, 0)\varphi(z)dz \\
&= e^{-rT}\int_{z_*}^{+\infty} (S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}z\right] - K, 0)\varphi(z)dz \\
&= S(0)e^{-rT+\left(r-\frac{1}{2}\sigma^2\right)T}\int_{z_*}^{+\infty} e^{\sigma\sqrt{T}z}\varphi(z)dz - Ke^{-rT}\int_{z_*}^{+\infty}\varphi(z)dz \\
&= S(0)e^{-\frac{1}{2}\sigma^2 T}\underbrace{\int_{z_*}^{+\infty} e^{\sigma\sqrt{T}z}\varphi(z)dz}_{A} - Ke^{-rT}\underbrace{\int_{z_*}^{+\infty}\varphi(z)dz}_{B}.
\end{aligned}$$

It is easy to compute the value of B:

$$\begin{aligned}
B &= \int_{z_*}^{+\infty}\varphi(z)dz \\
&= \mathbf{P}(z \geqslant z_*) \\
&= \mathbf{P}(z < -z_*) \\
&= \Phi(z_*).
\end{aligned}$$

where $\mathbf{P}(z \geqslant z_*)$ and $\mathbf{P}(z < -z_*)$ represent the probabilities that the standard normal random variable z is greater than or equal to, or less than z_*, respectively. In order to obtain the last equality we have used the symmetry properties of the p.d.f. of z; illustrated in figure 1.

We now focus on A:

$$\begin{aligned}
A &= \int_{z_*}^{+\infty} e^{\sigma\sqrt{T}z}\varphi(z)dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{z_*}^{+\infty} e^{\sigma\sqrt{T}z-\frac{1}{2}z^2}dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{z_*}^{+\infty} e^{-\frac{1}{2}(z^2-2\sigma z\sqrt{T}+\sigma^2 T)+\frac{1}{2}\sigma^2 T}dz \\
&= \frac{e^{\frac{1}{2}\sigma^2 T}}{\sqrt{2\pi}}\int_{z_*}^{+\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2}dz
\end{aligned}$$

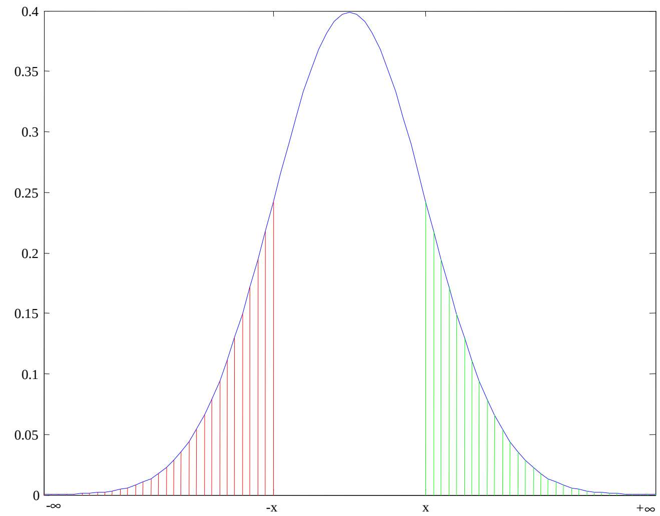

Figure 1: Diagram illustrating the symmetry of the p.d.f. for a standard normal variable. The area of the shaded region on the left is equal to the area of the shaded region on the right.

By using the change of variable $u = z - \sigma\sqrt{T}$ we get:

$$\begin{aligned} A &= \frac{e^{\frac{1}{2}\sigma^2 T}}{\sqrt{2\pi}} \int_{z_*-\sigma\sqrt{T}}^{+\infty} e^{-\frac{1}{2}u^2} du \\ &= e^{\frac{1}{2}\sigma^2 T} \mathbf{P}(z \geqslant z_* - \sigma\sqrt{T}) \\ &= e^{\frac{1}{2}\sigma^2 T} \mathbf{P}(z < -z_* + \sigma\sqrt{T}) \\ &= e^{\frac{1}{2}\sigma^2 T} \Phi(-z_* + \sigma\sqrt{T}). \end{aligned}$$

We now replace the values of A and B into the expression for $V(0)$:

$$V(0) = S(0)e^{-\frac{1}{2}\sigma^2 T} \underbrace{e^{\frac{1}{2}\sigma^2 T}\Phi(-z_* + \sigma\sqrt{T})}_{A} - Ke^{-rT} \underbrace{\Phi(z_*)}_{B}.$$

Finally, we obtain the celebrated Black-Scholes valuation formula for European calls:

$$V(0) = S(0)\Phi(-z_* + \sigma\sqrt{T}) - Ke^{-rT}\Phi(-z_*).$$

Since we know the expression for z_*, we can compute $-z_*$ and $-z_* + \sigma\sqrt{T}$ explicitly. Often, the Black-Scholes formulas are presented in a form in which the arguments of Φ

are given as follows:

$$\begin{aligned} V(0) &= S(0)\Phi(z_1) - Ke^{-rT}\Phi(z_2) \\ z_1 &= -z_* + \sigma\sqrt{T} = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right] \\ z_2 &= -z_* = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right]. \end{aligned}$$

0.2 Valuation of European Puts

We could compute the value of a European put following the same approach as above. However, we can use the put-call parity to save most of the work.

From now on, we will use the usual notation C for calls, and P for European puts. We immediately have:

$$\begin{aligned} C(0) - P(0) &= S(0) - Ke^{-rT} \\ -P(0) &= S(0) - C(0) - Ke^{-rT} \\ -P(0) &= S(0) - (S(0)\Phi(z_1) - Ke^{-rT}\Phi(z_2)) - Ke^{-rT} \\ -P(0) &= S(0)(1 - \Phi(z_1)) - Ke^{-rT}(1 - \Phi(z_2)). \end{aligned}$$

It is easy to rewrite expressions like $1 - \Phi(z)$ to a simpler form:

$$\begin{aligned} 1 - \Phi(z) &= 1 - \mathbf{P}(z' \leqslant z) \\ &= \mathbf{P}(z' > z) \\ &= \mathbf{P}(z' \leqslant -z) \\ &= \Phi(-z). \end{aligned}$$

This leads to the usual form for the price of the option:

$$P(0) = -S(0)\Phi(-z_1) + Ke^{-rT}\Phi(-z_2).$$

0.3 "The Greeks"

Since we have explicit formulas for computing the time-0 values of European puts and calls, we can study the sensitivity of the respective values to infinitesimal changes in the underlying parameters; these sensitivities are defined in table 1. Collectively, these five quantities are known as "the Greeks" (named so out of respect for the civilization that founded rational Western science, and not as a reminder of the fraternities' Greek system).

We note that while we are going to examine "the Greeks" only with respect to European puts and calls, these sensitivities can be also be defined and studied for other instruments or portfolios.

The Black-Scholes formula for the value of a call is

$$C = S\Phi(z_1) - Ke^{-rT}\Phi(z_2).$$

Name	**Definition**
delta	$\Delta = \frac{\partial V}{\partial S}$
gamma	$\Gamma = \frac{\partial^2 V}{\partial S^2}$
rho	$\rho = \frac{\partial V}{\partial r}$
theta	$\theta = \frac{\partial V}{\partial t}$
vega	$\upsilon = \frac{\partial V}{\partial \sigma}$

Table 1: The definition of sensitivities of values to infinitesimal changes in various parameters. Collectively, these quantities are known in the financial literature as "the Greeks."

(note that for brevity of notation we dropped the 0).

We might be tempted to conclude immediately that $\Delta_c = \frac{\partial C}{\partial S} = \Phi(z_1)$, because it seems that C depends explicitly only on the underlying stock price, through the presence of S itself in the valuation formula. Of course, this is not true, as z_1 and z_2 both depend on S. The answer is actually correct, but the correct computation is more convoluted than it appeared at first sight.

We have successively:

$$\begin{aligned}
\Delta_c &= \frac{\partial C}{\partial S} \\
&= \frac{\partial}{\partial S}\left(S\Phi(z_1) - Ke^{-rT}\Phi(z_2)\right) \\
&= \Phi(z_1) + S\frac{\partial \Phi}{\partial S}(z_1) - Ke^{-rT}\frac{\partial \Phi}{\partial S}(z_2).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\frac{\partial \Phi}{\partial S}(z) &= \frac{d\Phi}{dz}\frac{\partial z}{\partial S} \\
&= \varphi(z)\frac{\partial z}{\partial S} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\frac{\partial z}{\partial S}.
\end{aligned}$$

From the definition of z_1 and z_2 we get:

$$\frac{\partial z_1}{\partial S} = \frac{\partial z_2}{\partial S} = \frac{1}{\sigma S\sqrt{T}}.$$

We now return to the computation of delta:

$$\Delta_c = \Phi(z_1) + \frac{1}{\sigma\sqrt{2\pi T}}\underbrace{\left(e^{-\frac{1}{2}z_1^2} - \frac{K}{Se^{rT}}e^{-\frac{1}{2}z_2^2}\right)}_{E}.$$

It turns out that $E = 0$. This can be seen by writing the following series of equivalent equalities:

$$\begin{aligned}
e^{-\frac{1}{2}z_1^2} &= \frac{K}{Se^{rT}}e^{-\frac{1}{2}z_2^2}, \\
e^{\frac{1}{2}(z_2^2 - z_1^2)} &= \frac{K}{Se^{rT}}, \\
\frac{1}{2}(z_2^2 - z_1^2) &= \ln K - \ln S - rT, \\
-\frac{1}{2}(z_1 - z_2)(z_1 + z_2) &= \ln K - \ln S - rT, \\
-\frac{1}{2}\sigma\sqrt{T}\frac{1}{\sigma\sqrt{T}}2\left(\ln\frac{S}{K} + rT\right) &= \ln K - \ln S - rT, \\
\ln K - \ln S - rT &= \ln K - \ln S - rT.
\end{aligned}$$

The last equality is clearly true; for the formal proof of the claim that $E = 0$, you can read the series of equalities in reverse order.

We conclude that $\Delta_c = \Phi(z_1)$.

The put-call parity immediately furnishes the value of the delta for European puts:

$$\begin{aligned}
C - P &= S - Ke^{-rT} \\
\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} &= 1 \\
\Delta_p &= \Phi(z_1) - 1.
\end{aligned}$$

Let us now compute gamma for European calls:

$$\begin{aligned}
\Gamma_c &= \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta_c}{\partial S} \\
&= \frac{\partial \Phi}{\partial S}(z_1) \\
&= \varphi(z)\frac{\partial z_1}{\partial S} \\
&= \frac{\varphi(z)}{\sigma S\sqrt{T}} \\
&= \frac{1}{\sigma S\sqrt{2\pi T}}e^{-\frac{1}{2}z_1^2}.
\end{aligned}$$

The gamma for European puts is identical to that of calls with the same maturity and strike price:

$$\begin{aligned}
C - P &= S - Ke^{-rT} \\
\frac{\partial^2 C}{\partial S^2} - \frac{\partial^2 P}{\partial S^2} &= 0 \\
\Gamma_p = \Gamma_c &= \frac{\varphi(z)}{\sigma S\sqrt{T}}.
\end{aligned}$$

Name	Calls	Puts
Δ	$\Phi(z_1)$	$\Phi(z_1) - 1$
Γ	$\frac{\varphi(z)}{\sigma S\sqrt{T}}$	
ρ	$KTe^{-rT}\Phi(z_2)$	$-KTe^{-rT}\Phi(z_2)$
θ	$-\frac{\sigma\varphi(z_1)S}{2\sqrt{T}} - rKe^{-rT}\Phi(z_2)$	$-\frac{\sigma\varphi(z_1)S}{2\sqrt{T}} + rKe^{rT}\Phi(-z_2)$
υ	$\varphi(z_1)S\sqrt{T}$	

Table 2: Values for "the Greeks" for European calls and puts.

The full set of values for "the Greeks" of European puts and calls is given in table 2.

Let us now examine the evolution of Δ_c as a function of S (i.e. varying S only, and fixing all the other parameters). A typical dependency is shown in figure 2.

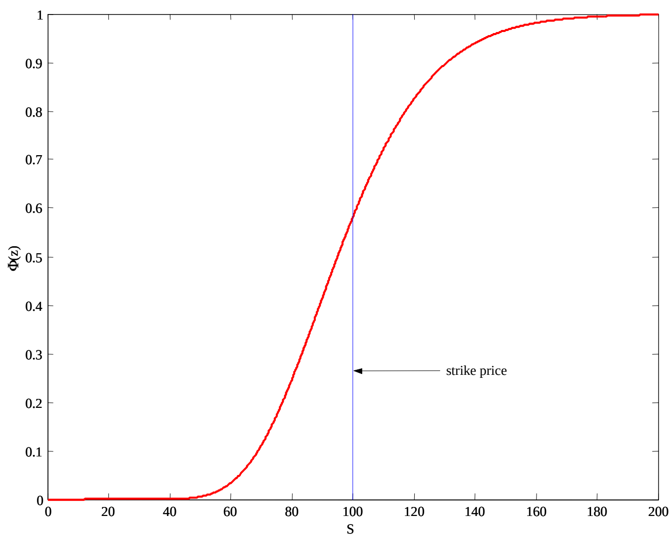

Figure 2: Variation of $\Delta_c = \Phi(z_1)$ for a European call. Recall that $z_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right]$. The values used to generate this graph are $r = .02$, $T = 1$, $K = 100$, $\sigma = .25$.

By examining figure 2, we immediately note that $\lim_{S\to\infty}\Delta_c = 1$. This means that for very large values of S, the call option on S changes in value almost exactly as much the underlying stock. But why is it so? Here is a qualitative argument:

If the stock price is very large w.r.t. K, then it is very likely that S will stay above K. If this is the case, the option will (almost) surely be exercised. In effect, holding the option is equivalent to a mandatory agreement to buy the underlying share at price K at expiration time T; such an agreement is called a forward contract. Based on a simple arbitrage argument the price of such a forward contract must be $S - Ke^{-rT}$, where S is the current price of the underlying stock.[3]Thus we have the approximate equality:

$$\lim_{S\to\infty} C \approx S - Ke^{-rT}.$$

Note that we are using the notation $\lim_{S\to\infty}$ loosely, to indicate a very large price for the underlying, not a true limit computation.

Given the approximation above, it is easy to see that $\lim_{S\to\infty} \Delta_c = 1$, as we have established previously.

What about the situation when the underlying stock price decreases toward 0? We have established before, when we talked about the bounds for option prices, that if at any time the price of a stock reaches 0, then it will always stay 0. This is true, in essence, because the price of the stock represents the discounted value of its future cash flows; if such cash flows are 0 now, they will also be 0 in the future. Now, if $S = 0$, then the call will never be exercised, and thus $C = 0$, which immediately implies that $\Delta_c = 0$.

Argue that this last equality must hold approximately for very small - but non-zero - values of S! Also, try to follow a similar line of reasoning for calls. Hint: The results can most easily be obtained by using the put-call parity.

0.4 Hedging

Except, perhaps, for invesments in shortest-term Treasuries, all investments involve risks (e.g. default risk, interest rate risk, inflation risk). Unfortunately, the return on short-term Treasuries is paltry, and does not satisfy the needs of most investors; hence, these investors are continuously looking for better opportunities. Higher returns, however, are also associate with higher risks. Put simply, if it is possible to achieve gains, then losses are also possible. Even worse, in general the possibility of big gains is associated with the possibility of big losses.

While gains are rewarding, individuals and institutions exhibit a high degree of loss aversion, both for psychological and pragmatic reasons. Think, for example, of a pension fund that must honor certain obligations at various future moments of time; its managers would likely want to avoid situations when the value of their holdings decreases significantly enough that these obligations can not be honored. If such an event occurs, their investors will not be too much consoled by the fact that the fund had actually hoped to increase the value of their assets by undertaking risky investments, but failed.

The considerations above imply that many investors are very motivated to manage the risks they assume by undertaking various investements. They are desirious of eliminating (or mitigating) the risk of downside from their portfolios, even if this means that they also must give up some of the potential for the corresponding upside. Often, investors would like their portfolios to be effectively immune to certain random changes that could occur in the economy, e.g. to changes in the level of interest rates.

[3] You can easily convince yourselves that this is true by constructing a portfolio that will reproduce the payoff of the option at expiration $S(T) - K$. The time-0 value of this portfolio is $S(0) - Ke^{-rT}$.

0.4.1 Delta-Hedging

Let us consider a portfolio $\mathbf{P}$, whose value depends, among other parameters, on the value of an underlying instrument S. Portfolio $\mathbf{P}$ is given; it resulted as a series of transactions in the past. In the extreme case, $\mathbf{P}$ might consist of a single instrument, but $\mathbf{P}$ might also be a complex collection of diverse instruments. Our goal is to extend P with f units of a new instrument, denoted D, so that the new portfolio becomes immune to infinitesimal changes in the underlying S. We get:

$$\begin{aligned} &V = \mathbf{P} + fD \\ &\frac{\partial V}{\partial S} = \underbrace{\frac{\partial \mathbf{P}}{\partial S}}_{\Delta_P} + f \underbrace{\frac{\partial D}{\partial S}}_{\Delta_D} = 0. \end{aligned}$$

Thus the number of units of D needed to immunize $\mathbf{P}$ is equal to

$$f = -\frac{\Delta_P}{\Delta_D}.$$

Let us assume that D is the underlying itself, i.e. $D = S$, and $\mathbf{P}$ is a call. We then get

$$f = -\Delta_C = -\Phi(z_1).$$

The last relation implies that in order to hedge a call, we must always short a fraction of the underlying stock. To start understanding this relationship, let us examine first the situation when the price of the underlying is so large that the call is, in effect, equivalent to a forward contract with value $S - Ke^{-rT}$. In this case $\Delta_C \approx 1$, thus we must "add" $f = -1$ units of stock to our portfolio. The extended portfolio will have a value of approximately $S - Ke^{-rT} - S = -Ke^{-rT}$, which is immune to changes in the underlying, within the precision of our approximation.

In practice, delta-hedging will not work perfectly, due to the following causes:

1. Our models are necessarily simplifications of reality, thus even an accurate computation of delta within the model will produce, in general, a value that is not equal to the true delta.

2. Even if our models were perfectly accurate, our computations are not.

3. We assumed only an infinitesimal change in the underlying's price; in practice, changes are necessarily finite.

4. Hedging must occur continuously, i.e. it must be done on intervals of time of infinitesimal length. Again, this is impossible in practice.

5. We have ignored transaction costs. For hedging operations that involve (infinitely) frequent trades, this is clearly not reasonable.

For all the reasons listed above, we have to accept that delta-hedging will inherently be approximate in any practical setting. An approximate hedge, however, is still far better than not having a hedge at all.

Here is how practical hedging can be implemented:

1. Initially, one must compute f, and buy f units of D, and add them to the portfolio. Any money needed for this must be borrowed from the money market account.

   Note: Strictly speaking, "buy" must be replaced with "sell," and "borrow" must be replaced with "lend," if $f < 0$. A similar issue arises for the finite changes in f, Δf which induces further buying if $\Delta f > 0$, or selling, if $\Delta f < 0$.

2. After a short period of time, we compute the new value of f, and the change of f from the time of the last rebalancing operation, Δf. We then buy (sell) Δf units of D, and borrow from (or return money into) the money market account. We repeat this procedure as often as practical under the circumstances.

0.4.2 Gamma-Hedging

One reason for the imperfect functioning of the delta-hedge, as we pointed out above, is that in practice we have to deal with finite (as opposed to infinitesimal) price changes and finite-length intervals. One problem is that deltas themselves change as the price of the underlying changes. This observation can be used to build portfolios that are immune to infinitesimal changes in delta induced by changes in the price of the underlying. We thus need to create portfolios whose gamma is equal to 0. We can achieve this by extending the original portfolio $\mathbf{P}$ with two instruments D_1, and D_2. We then have:

$$V = \mathbf{P} + f_1 D + f_2 D$$

$$\begin{cases} \frac{\partial V}{\partial S} = \underbrace{\dfrac{\partial \mathbf{P}}{\partial S}}_{\Delta_{\mathbf{P}}} + f_1 \underbrace{\dfrac{\partial D_1}{\partial S}}_{\Delta_{D_1}} + f_2 \underbrace{\dfrac{\partial D_2}{\partial S}}_{\Delta_{D_2}} = 0 \\ \frac{\partial^2 V}{\partial S^2} = \underbrace{\dfrac{\partial^2 \mathbf{P}}{\partial S^2}}_{\Gamma_{\mathbf{P}}} + f_1 \underbrace{\dfrac{\partial^2 D_1}{\partial S^2}}_{\Gamma_{D_1}} + f_2 \underbrace{\dfrac{\partial^2 D_2}{\partial S^2}}_{\Gamma_{D_2}} = 0 \end{cases}$$

$$\begin{cases} \Delta_{\mathbf{P}} + f_1 \Delta_{D_1} + f_2 \Delta_{D_2} = 0 \\ \Gamma_{\mathbf{P}} + f_1 \Gamma_{D_1} + f_2 \Gamma_{D_2} = 0 \end{cases}.$$

The last system of equations can be solved to obtain the values of f_1 and f_2.

Like we did before, we can assume that one of the instruments that is used for hedging is the underlying itself. Let us assume that $D_2 = S$. We now obtain a simpler form for the system of equations:

$$\begin{cases} \Delta_{\mathbf{P}} + f_1 \Delta_{D_1} + f_2 = 0 \\ \Gamma_{\mathbf{P}} + f_1 \Gamma_{D_1} = 0 \end{cases}.$$

This simple form allows us to construct a gamma-hedged portfolio in two simple steps. First, we use the second equation to determine the number of units of D_1 that we need to make the composite portfolio gamma-neutral; $f_1 = -\frac{\Gamma_P}{\Gamma_{D_1}}$. Adding the underlying to this portfolio does not make any difference, as $\Gamma_2 = \Gamma_s = 0$. We can then determine the number of units of the underlying needed to make the portfolio delta-neutral; $f_2 = -\Delta_P - f_1\Delta_{D_1}$.

The simplified procedure above is possible because $\Gamma_s = 0$. Such an equality also holds for the money market account B: $\Gamma_B = 0$. Can we use the money market account to hedge our portfolio? The system of equations is reduced to the following:

$$\begin{cases} \Delta_{\mathbf{P}} + f_1\Delta_{D_1} + f_2 0 = 0 \\ \Gamma_{\mathbf{P}} + f_1\Gamma_{D_1} + f_2 0 = 0 \end{cases}.$$

The two equations provide the value of f_1 independently, to be $-\frac{\Delta_{\mathbf{P}}}{\Delta_{D_1}}$, and $-\frac{\Gamma_{\mathbf{P}}}{\Gamma_{D_1}}$, respectively. As in general these two quantities are not equal, hedging with the money market account is not possible. This is ultimately due to the fact that $\Delta_B = \Gamma_B = 0$.

Finally, we note that gamma-hedging, while more resilient than simple delta-hedging, is still approximate for finite changes in the underlying's price.

0.4.3 Computational Issues

As pointed out above, "the Greeks" can be defined for other instruments than options, and also for portfolios. For the case of European puts and calls in the Black-Scholes model we are in the fortunate situation in which we have analytic valuation formulas, which can then be used to determine the values for "the Greeks."

In general, however, we do not have closed-form formulas. Can we still compute "the Greeks" under these circumstances? Yes.

In fact, all we need to do is to employ the techniques we have discussed before. If we assume that we have a numerical method to compute the value of a given instrument or portfolio. Let us denote the value returned by this procedure with $\mathbf{V}(t, S, r, \sigma)$. We have made explicit the dependency of $\mathbf{V}$ on the parameters relevant for "the Greeks;" of course, $\mathbf{V}$ might depend on other parameters as well. We can then compute approximate values for Δ and Γ, for example, by using the formulas below:

$$\begin{aligned} \Delta &\approx \frac{\mathbf{V}(t, S + h_s, r, \sigma) - \mathbf{V}(t, S - h_s, r, \sigma)}{2h_s} \\ \Gamma &\approx \frac{\mathbf{V}(t, S + h_s, r, \sigma) - 2\mathbf{V}(t, S, r, \sigma) + \mathbf{V}(t, S - h_s, r, \sigma)}{h_s^2}. \end{aligned}$$

Approximate values for the other "Greeks" can be computed similarly.

All techniques (e.g. Richardson's extrapolation), comments, and observations that we made when discussing numerical differentiation in general still hold. We just note here that if an instrument is very insensitive to the change in S, for example, then the expressions that define Δ and Γ might suffer from cancellation errors; their value will be determined mostly by accumulated computational errors, and will not reflect the true values.

0.5 The Black-Scholes Differential Equation

0.5.1 Stochastic Differential Equation for the Evolution of S

For this section, you will have to recall our discussion on the evolution of stock prices. We can summarize our earlies conclusions as follows ($r_T = \log \frac{S(T)}{S(0)}$, the interval under examination is $[0, T]$):

$$\begin{cases} r_T \ is \ normal \\ E[r_t] = \mu T \\ Var[r_t] = \sigma^2 T \end{cases}.$$

These statements imply that the stock prices themselves are log-normally distributed. We can summarize our conclusions by writing:

$$r_T = \log \frac{S(T)}{S(0)} = \mu T + \sigma z \sqrt{T},$$

where z is a standard normal random variable.

Assume now that the length of the interval $[0, T]$ decreases toward 0:

$$\begin{aligned} \log \frac{S(T)}{S(0)} &= \log \left(1 + \frac{S(T) - S(0)}{S(0)}\right) \\ &= \log \left(1 + \frac{\Delta S}{S(0)}\right) \\ &\approx \frac{\Delta S}{S(0)}. \end{aligned}$$

The last step is justified by the assumption that as the length of the interval decreases toward 0, the change of the stock price that corresponds to the respective interval decreases. We then used the approximation that for small x, $\log(1 + x) \approx x$.[4]

When T becomes very small, we can write $\frac{\Delta S}{S(0)}$ as $\frac{dS}{S}$, to emphasize this smallness; also, we can write $T = dt$.

The quantity $\sigma z \sqrt{T}$, however, does not necessarily decrease toward 0 as T becomes very small. This is because z is a standard random variable whose magnitude does not depend on the length of the interval. As we decrease the length of the interval, the probability of getting a small value for $\sigma z \sqrt{T}$ becomes very large, but there will always be a non-zero probability for $\sigma z \sqrt{T}$ to be arbitrarily large.[5] This is true because z can take values that are arbitrarily large, and the magnitude of z can compensate for the smallness of T.

The quantity that we associate with $z\sqrt{T}$ is traditionally examined by identifying it with the increments over $[0, T]$ of a stochastic process, named the Wiener process. Introducing the notation $W(t)$ for the process itself, and $dW(t)$ for its increments, we

[4] You can verify this if you consider a Taylor-series expansion of $\log(1 + x)$ around 0.

[5] Note that if $z\sqrt{T}$ is large, then our approximation for the natural logarithm will not hold. We are glossing over some details in order to focus on the main issues.

can write the relationship that expresses the random changes in the stock price over infinitesimal intervals:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

Note that in the formula above we have made explicit the dependence of S and W on time. We will not use this explicit notation systematically, but we should be aware such a dependency exists.

The Wiener process has very important practical applications and it is studied extensively in courses on stochastic calculus. We list some its properties without further proof:

1. $W(t)$ is continuous at every point, but not differentiable at any point.

2. No matter where is starts, W will ultimately hit all points. Once W hits a point, it will hit it infinitely often immediately after that.

3. The increments ΔW of W on any time interval Δt are normally distributed, such that $\mathbb{E}[\Delta W] = 0$, $Var[\Delta W] = \Delta t$.

4. The increments of W on non-overlapping intervals are independent.

0.5.2 Stochastic Differential Equation for the Evolution of a European Payoff's Value

Assume that we have a European instrument whose value $V = V(t, S)$ depends on time, and on the value of an underlying S. The value might depend on other parameters as well (for example, on volatility), but these parameters will be assumed to be constant. Further, assume that the infinitesimal changes in S are given by $dS = Adt + BdW$, where W is a standard Wiener process.

Assuming for a moment that dW is a regular (i.e. non-random) differential. Using the rules of differential calculus we get the following equalities:

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS \\ dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}(Adt + BdW) \\ dV &= B\frac{\partial V}{\partial S}dW + \left(\frac{\partial V}{\partial t} + A\frac{\partial V}{\partial S}\right)dt. \end{aligned}$$

Due to the unusual nature of the Wiener process the last equality derived above does not hold. Without further proof, we will provide the correct result, which is

$$dV = B\frac{\partial V}{\partial S}dW + \left(\frac{\partial V}{\partial t} + \frac{1}{2}B^2\frac{\partial^2 V}{\partial S^2} + A\frac{\partial V}{\partial S}\right)dt.$$

This result is known as (a particular case of) Itô's lemma. Due to the randomness in the Wiener process, a new term had to be added to establish equality.

By introducing the known values for A and B in the formula from Itô's lemma, we obtain the following:

$$dV = \sigma S \frac{\partial V}{\partial S} dW + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} \right) dt.$$

0.5.3 Eliminating Randomness

Consider a portfolio that consists of one unit of V, and Δ units of the underlying S, where Δ is a fixed number left unspecified for now. We get:

$$\mathbf{P} = V - \Delta S$$

$$d\mathbf{P} = dV - \Delta dS.$$

By replacing the known expressions for dV and dS in the equality above, we obtain that

$$dV = \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dW + \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \left(\frac{\partial V}{\partial S} - \Delta \right) \right] dt.$$

By choosing Δ to be equal to $\frac{\partial V}{\partial S}$ (which is exactly the way we defined Δ above, as the sensitivity of the value to infinitesimal changes in the price of the underlying), the formula simplifies to

$$d\mathbf{P} = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

The last equality is remarkable because it contains no random component; all randomness has been eliminated due to the clever choice of Δ. Note that Δ is fixed on each infinitesimal interval, but it changes from one such interval to the next one. This idea is akin to the continuous rebalanced delta-hedging that we studied above.

Since the quantity $d\mathbf{P}$ is deterministic, its magnitude can be related to the amount of money that would be earned on the money market account if an amount equal to $\mathbf{P}$ were invested for an infinitesimal period dt. It turns out that the two quantities must be equal: $d\mathbf{P} = r\mathbf{P}dt$.

The justification of the last relation is immediate, and it is based on arbitrage considerations. Indeed, assume for a moment that $d\mathbf{P} > r\mathbf{P}dt$. As both quantities in this expression are known at the beginning of the interval - and they are deterministic - we could choose to borrow money from the money market account and invest it in our portfolio. If we did this, we could earn with certainty an amount $d\mathbf{P} - r\mathbf{P}dt > 0$. This is a clear case of arbitrage! A similar reasoning will convince us that it is not possible to have $d\mathbf{P} < r\mathbf{P}dt$.

By replacing the expression for $d\mathbf{P}$ and $\mathbf{P}$ in the equality $d\mathbf{P} = r\mathbf{P}dt$, we get:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r \left(V - S \frac{\partial V}{\partial S} \right).$$

We have finally obtained the celebrated Black-Scholes differential equation for European options:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

By introducing the operator $\mathbf{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r$, we can rewrite the equation in the form $\mathbf{L}V = 0$.

Such a differential equation can not be solved fully without imposing a set of conditions on it.

0.5.4 Conditions for European Calls

For specificity, we replace $V(t, S)$ by $C(t, S)$. We obtain the following conditions:

$$\begin{aligned} C(T, S) &= \max(S - K, 0) \\ C(t, 0) &= 0 \\ \lim_{S\to\infty} C(t, S) &= S - Ke^{-r(T-t)}. \end{aligned}$$

Again, note that we are using the limit notation loosely, to indicate that the equality holds approximately for very large values of S.

0.5.5 Conditions for European Puts

By denoting $V(t, S)$ with $P(t, S)$, we obtain:

$$\begin{aligned} P(T, S) &= \max(K - S, 0) \\ P(t, 0) &= Ke^{-r(T-t)} \\ \lim_{S\to\infty} P(t, S) &= 0. \end{aligned}$$

These relations can be obtained from the put-call parity. It is interesting to note that if $S = 0$ at a certain moment of time then the put will be surely exercised (we know this because S will be 0 for all future moments of time as well). If the put will be exercised for sure, its value must be the present value at time t of its strike price K received at time T. This is an alternative proof for the second equality above.

0.6 Reduction to the Heat Equation

The Black-Scholes differential equation, as written, is a backward-parabolic equation.[6] Directly solving this equation is not easy, one reason being that conditions are given not for the initial time $t = 0$, but for the expiration time $t = T$. Rather than "propagating" the initial conditions toward the future, we need to "bring back" the final conditions toward the past. This is not easy to do. To avoid these difficulties we transform the

[6] For details, see section 1.2 of the handout titled "Partial Differential Equations."

Black-Scholes differential equation into a forward equation. By using a series of suitable substitutions, in fact, we will achieve even more: we will simplify the form of the equation by reducing it to the heat equation.

0.6.1 Transformation to a Forward Equation

We perform the following substitutions

$$\begin{cases} S = Ke^x \\ t = T - \frac{1}{\frac{1}{2}\sigma^2}\tau \\ V(t,S) = Kv(\tau, x) \end{cases}.$$

in the Black-Scholes differential equation given below:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

We start by expressing all the terms that appear in the differential equation in terms of the new variables.

$$\begin{aligned} \frac{\partial V}{\partial t} &= K\frac{\partial v}{\partial t}(\tau, x) \\ &= K\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} \\ &= -\frac{1}{2}\sigma^2 K\frac{\partial v}{\partial \tau}. \end{aligned}$$

Here is the computation for the first order partial derivative with the respect to the price of the underlying S:

$$\begin{aligned} \frac{\partial V}{\partial S} &= K\frac{\partial v}{\partial S}(\tau, x) \\ &= K\frac{\partial v}{\partial x} \\ &= \frac{K}{S}\frac{\partial v}{\partial x}. \end{aligned}$$

We can now compute the second derivative:

$$\begin{aligned} \frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{K}{S}\frac{\partial v}{\partial x}\right) \\ &= -\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S}\frac{\partial^2 v}{\partial x^2}\frac{\partial x}{\partial S} \\ &= \frac{K}{S^2}\left(-\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right) \\ &= \frac{1}{Ke^{2x}}\left(-\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right). \end{aligned}$$

Old Condition	**Transformed Condition**
$C(t,0)=0$	$\lim_{x\to-\infty} C_v(\tau,x)=0$
$\lim_{\lim_{x\to\infty}} C(t,S)=S-Ke^{-r(T-t)}$	$\lim_{x\to\infty} C_v(\tau,x)=e^x-e^{-k\tau}$
$C(T,S)=\max(S-K,0)$	$C_v(0,x)=\max(e^x-1,0)$

Table 3: Original, and transformed conditions for European calls. Note: We employed the notation C_v for the transformed European call value function.

We now replace these terms in the Black-Scholes equation:

$$-\frac{1}{2}\sigma^2 K\frac{\partial v}{\partial \tau}+\frac{1}{2}\sigma^2 S^2\frac{K}{S^2}\left(-\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}\right)+rS\frac{K}{S}\frac{\partial v}{\partial x}-rKv=0$$

$$-\frac{1}{2}\sigma^2\frac{\partial v}{\partial \tau}+\frac{1}{2}\sigma^2\left(-\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}\right)+r\frac{\partial v}{\partial x}-rv=0$$

$$-\frac{\partial v}{\partial \tau}-\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}+\frac{r}{\frac{1}{2}\sigma^2}\frac{\partial v}{\partial x}-\frac{r}{\frac{1}{2}\sigma^2}v=0$$

$$-\frac{\partial v}{\partial \tau}+\left(\frac{r}{\frac{1}{2}\sigma^2}-1\right)\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}-\frac{r}{\frac{1}{2}\sigma^2}v=0.$$

If we introduce the notation $k=\frac{r}{\frac{1}{2}\sigma^2}$, we get:

$$-\frac{\partial v}{\partial \tau}+(k-1)\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}-kv=0.$$

We must not forget to rewrite the conditions so that they correspond to the new variables.

$$x=\ln\frac{S}{K}\Rightarrow\begin{cases} S\to 0\Rightarrow x\to-\infty \\ S\to\infty\Rightarrow x\to\infty\end{cases}$$

$$\tau=\frac{1}{2}\sigma^2(T-t)\Rightarrow\begin{cases} t\to 0\Rightarrow \tau=\frac{1}{2}\sigma^2 T \\ t\to T\Rightarrow \tau=0\end{cases}.$$

The upper and lower boundaries are now at $\pm\infty$. More importantly, the "arrow of time" has been reversed. By introducing τ, the time in the transformed equation flows backwards versus the time in the initial equation. This has the advantage that the end condition that we had before can be transformed into an initial condition. For calls, we obtain conditions given in table 3.

Try to derive analogous formulas for European puts!

0.6.2 Reduction to the Heat Equation

At this point, we have succeeded in transforming the differential equation to a forward equation, i.e. we have conditions at $t=0$, and at the boundaries $x=-\infty$ and $x=\infty$. The form of the differential equation, however, is still not simple enough for our purposes.

We now proceed to reduce the equation to the heat equation. Leaving - for the moment - quantities α and β to be indeterminate, we perform the following change of function:

$$v(\tau, x) = e^{\alpha x + \beta\tau} u(\tau, x).$$

We now determine the form of the terms that appear in the differential equation:

$$\begin{aligned}
\frac{\partial v}{\partial \tau} &= \left(\beta u + \frac{\partial u}{\partial \tau}\right) e^{\alpha x + \beta\tau} \\
\frac{\partial v}{\partial x} &= \left(\alpha u + \frac{\partial u}{\partial x}\right) e^{\alpha x + \beta\tau} \\
\frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}\left[\left(\alpha u + \frac{\partial u}{\partial x}\right) e^{\alpha x + \beta\tau}\right] \\
&= \left[\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \alpha\left(\alpha u + \frac{\partial u}{\partial x}\right)\right] e^{\alpha x + \beta\tau} \\
&= \left(\alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right) e^{\alpha x + \beta\tau}.
\end{aligned}$$

After replacing the terms in the differential equation and dividing by $e^{\alpha x + \beta\tau}$, we get:

$$-\left(\beta u + \frac{\partial u}{\partial \tau}\right) + \left(\alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right) + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku = 0.$$

$$-\frac{\partial u}{\partial \tau} + (2\alpha + k - 1)\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \left[-\beta + \alpha(k-1) + \alpha^2 - k\right] u = 0.$$

We eliminate the term containing u by requiring that $-\beta + \alpha(k-1) + \alpha^2 - k = 0$. This gives us a value for β in terms of α and k:

$$\beta = \alpha(k-1) + \alpha^2 - k.$$

Setting $2\alpha + k - 1 = 0$, we can also eliminate the term containing $\frac{\partial u}{\partial x}$. We obtain:

$$\begin{aligned}
\alpha &= -\frac{1}{2}(k-1) \\
\beta &= -\frac{1}{2}(k-1)^2 + \frac{1}{4}(k-1)^2 - k = -\frac{1}{4}(k+1)^2.
\end{aligned}$$

With these values for α and β, we can determine the actual substitution needed to reduce the differential equation from the intermediate stage to the heat equation:

$$v(\tau, x) = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} u(\tau, x).$$

With this substitution, we have shown that the Black-Scholes equation reduces to the heat equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

We must, of course, also rewrite the conditions so that they correspond to the variable transformation that we have employed. By using C_u to denote the function giving the value of the European call in the variables that correspond to the last substitution, we obtain the following set of conditions:

$$\begin{aligned}\lim_{x\to-\infty} C_u(\tau, x) &= 0 \\ \lim_{x\to\infty} C_u(\tau, x) &= e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k+1)^2\tau}\left(e^x - e^{-k\tau}\right) \\ C_u(0, x) &= \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0).\end{aligned}$$

Again, you should try to derive analog formulas for European puts.

0.6.3 Computational Issues

It is possible to analytically solve the heat equation with the conditions that we have specified above. We can then undo the variable transformations and find the solution in the original variables. As we already know the Black-Scholes valuations formulas from an alternative proof, we do not need to do that again. Instead, we focus on the numerical solution. The insights we gain here can later be generalized to problems that do not admit closed-form solutions.

We have discussed extensively how to solve the heat equation; and we have described in detail three methods: the explicit method, the fully implicit method, and the Crank-Nicholson method. From among the methods studied we recommended Crank-Nicholson, because it is both precise (it has an error term $O((\delta t)^2) + O((\delta x)^2)$), and unconditionally stable[7] (i.e. we can choose any values for $\alpha = \frac{\delta t}{(\delta x)^2}$ without worrying that small perturbations in the solution will grow unboundedly).

We also know how to handle conditions at $\pm\infty$; we just replace the infinite values with suitably chosen value $N_{\min}$ and $N_{\max}$, which are "good" approximations of infinity in the context of our problem.

In principle, we are done. In practice, finite difference methods suffer from a subtle problem induced by the changes of variable that we performed, and by the requirement to use a regularly spaced grid in the space of the transformed variables.

Let us recall the variable changes that we undertook:

$$\begin{aligned} S &= Ke^x \\ t &= T - \frac{1}{\frac{1}{2}\sigma^2}\tau. \end{aligned}$$

When solving the heat equation, we are discretizing the domain in the x and τ variables. Let us assume that the grid steps along the space and time dimension are δx, and

[7] Note that stability is only one precondition for us to get a solution close to the true one. Accuracy is also required. While in the limit, when both $\delta t \to 0$ and $\delta x \to 0$, Crank-Nicholson will converge to the true solution, the choice finite values for δt and δx (and - implicitly - for α) will influence the quality of the solution we get.

$\delta\tau$, respectively. These steps correspond to changes in the variables S and t as well:

$$\begin{aligned} S_{k+1} &= Ke^{x_{k+1}} = Ke^{x_k+\delta x} = S_k e^{\delta x} \\ t_{k+1} &= t_k - \frac{1}{\frac{1}{2}\sigma^2}\delta\tau. \end{aligned}$$

While the regular grid along the transformed time dimension induces a regular grid in the original time dimension, the regular grid in the transformed space dimension induces an irregular grid in the original space dimension. It is the ratio $\frac{S_{k+1}}{S_k}$ that is constant, not the difference $S_{k+1} - S_k$! More, as $\delta\tau > 0$, this ratio is superunitary, which means that the length of intervals $S_{k+1} - S_k$ monotonically increases toward the right end of the interval $[S_{\min}, S_{\max}]$.

The relation between the grids in the original and the transformed coordinates is illustrated in figure 3.

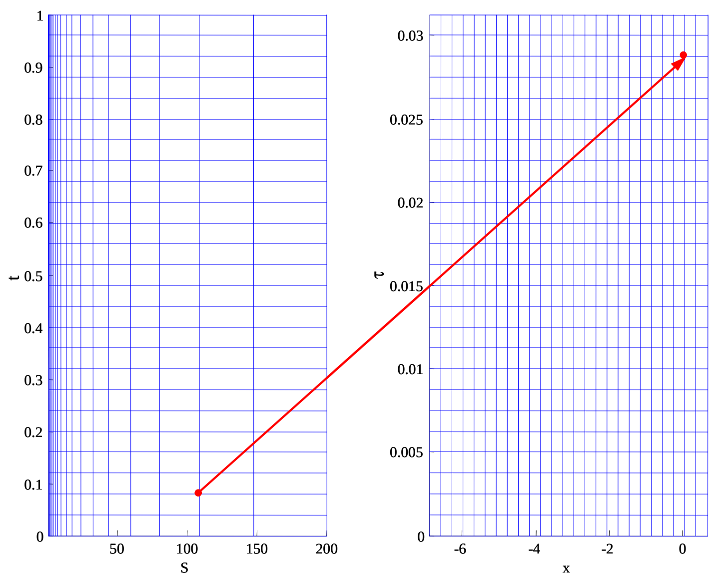

Figure 3: Relationship between the regular grid in the coordinate system (τ, x) of the transformed equations, and the introduced irregular grid in the original system of coordinates (t, S). The values shown here have been obtained for $S \in [.1, 200]$, $K = 100$, $T = 1$, $\sigma = .25$. There are 25 intervals along both dimensions in both grids. Two corresponding points are shown explicitly.

Our insights into the irregularity of the grid in the original coordinates must inform the way in which we structure our computations. If we want to sample the solution of the Black-Scholes equation with a step in the original space coordinate not larger than, say, ΔS, we must choose the step in the transformed system of coordinates accordingly.

Let us assume that $S \in [S_{\min}, S_{\max}]$; this means that the transformed coordinate x will be in the interval $[\log \frac{S_{\min}}{K}, \log \frac{S_{\max}}{K}]$. Further, assume that we divide the domain of variation of x into N_x subintervals of length δx each. This division also induces N_x intervals on the original coordinate S.

The largest induced step in the original space coordinate (S) occurs on the last subinterval of x. Using notation analogous to that introduced when studying finite difference methods, we have:

$$S_k = Ke^{x_k},\ k = \overline{0, N_x},\ x_k = x_{\min} + k\delta x.$$

Now we can write the following:

$$\begin{aligned} S_{N_x} - S_{N_x - 1} = Ke^{x_{\max}} - Ke^{x_{\max} - \delta x} &\leqslant \Delta S \\ \underbrace{Ke^{x_{\max}}}_{S_{\max}}(1 - e^{-\delta x}) &\leqslant \Delta S. \end{aligned}$$

After some simple algebraic manipulation, we get:

$$\delta x \leqslant -\log\left(1 - \frac{\Delta S}{S_{\max}}\right) \approx \frac{\Delta S}{S_{\max}}.$$

Let us assume that we choose the highest value for δx consistent with the inequality above. For simplicity, we will assume that this value leads to an integer number of intervals. Let us compute the smallest interval induced by our choice of δx in the original space coordinate. This smallest interval corresponds to the following difference:

$$\begin{aligned} S_1 - S_0 &= Ke^{x_{\min} + \delta x} - Ke^{x_{\min}} \\ &= \underbrace{Ke^{x_{\min}}}_{S_{\min}}(e^{\delta x} - 1) \\ &= S_{\min}\left[e^{-\log\left(1 - \frac{\Delta S}{S_{\max}}\right)} - 1\right] \\ &= \frac{S_{\min}}{S_{\max} - \Delta S}\Delta S \\ &\approx \frac{S_{\min}}{S_{\max}}\Delta S. \end{aligned}$$

In order for $x_{\min}$ and $x_{\max}$ to be "good approximations" of $\pm\infty$, we must choose $S_{\min}$ to be close to 0, and $S_{\max}$ to be much higher than K. Let us pick $S_{\min} = .1$, and $S_{\max} = 200$ (for a strike price of, say, $K = 100$). If the largest induced interval in the original space coordinate will be of length no greater than ΔS, then the smallest induced interval will be approximately of length $.0005\Delta S$. The ratio between the smallest and longest interval in the original coordinate S is $1 : 2000$ for this example! Clearly, we will end up doing a lot of unnecessary work. This observation points to a clear limitation of finite difference methods.

Ideally, we would like create a regular grid in the original coordinates (t, S), and then use a method that can handle an irregular grid in the transformed coordinates. Such methods exist, and are known as finite element methods. We will not study finite element methods in this class.

Using finite differences we obtain the approximate values of the solution only at a set of discrete points (the grid nodes). If we are also interested in the value of the examined option at intermediate points as well, we can use interpolation methods to obtain an approximate value based on the values associated with the neighboring grid points.